Problem for 2014 June

Communicated by Dan Jurca

The following problem is from *The Red Book of Mathematical Problems* by Kenneth S. Williams and Kenneth Hardy.

Let R be a finite ring containing an element r which is not a divisor of zero. Prove that R must have a multiplicative identity.

Solution by Dan Jurca

Suppose the (finitely many) elements of R are $r_1,\ r_2,\ r_3,\ \ldots, r_n$, so that r is one of the r_k. If $rr_i = rr_j$, then $rr_i - rr_j = 0_R$ (where 0_R is the additive identity in R). Hence $r(r_i - r_j) = 0_R$, and since by hypothesis r is not a zero divisor, it follows that $r_i - r_j = 0_R$, so $r_i = r_j$. Thus the n elements $rr_1,\ rr_2,\ rr_3,\ \ldots, rr_n$ are distinct, from which it follows that for some $i_0,\ 1 \le i_0 \le n,\ rr_{i_0} = r$. Then we show $r_{i_0} = 1_R$, the multiplicative identity in R.

For if $r_{i_0} r_i = r_j$, then $r(r_{i_0} r_i) = rr_j$, so $(rr_{i_0})r_i = rr_i = rr_j$, whence $r_i = r_j$. Thus $1 \le i \le n \Rightarrow r_{i_0} r_i = r_i$, and r_{i_0} is a left multiplicative identity.

A similar argument based on the observation that $r_1 r,\ r_2 r,\ r_3 r,\ \ldots, r_n r$ are distinct shows that there exists a right multiplicative identity, say r_{i_1}. But then $r_{i_1} = r_{i_0} r_{i_1} = r_{i_0}$, so that $r_{i_0} = r_{i_1}$ is a two-sided multiplicative identity in R.

Remark. One observes that the subring $2\mathbf{Z}$ of $\mathbf{Z}$, the integers, is an infinite ring with no nonzero zero divisor, and $2\mathbf{Z}$ does not contain a multiplicative identity.

Also solved by Arthur Fabian, Massoud Malek, and Winston Teitler