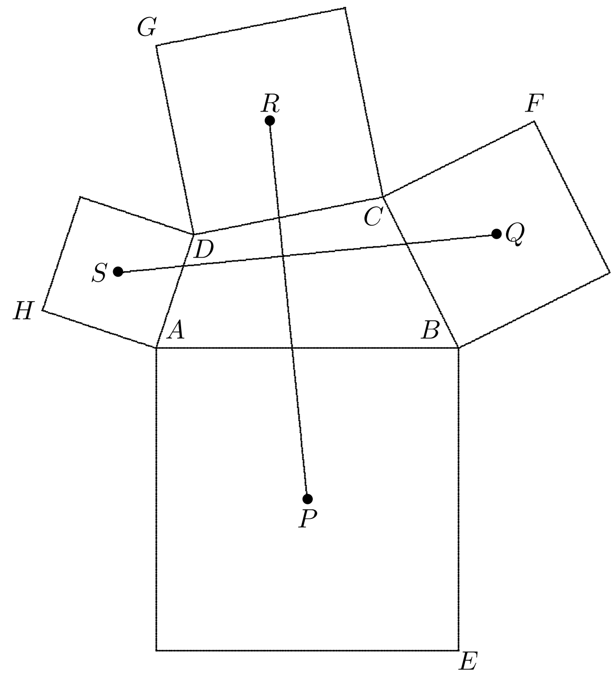
Problem for 2014 October

Communicated by Dan Jurca

The vertices of an arbitrary quadrilateral are labeled counterclockwise A, B, C, and D; on each side a square is constructed as in the sketch below. Point P is the center of the square on side AB; point Q is the center of the square on side BC; point R is the center of the square on side CD; and point S is the center of the square on side DA. Prove that segments PR and QS are of equal length and are perpendicular.



Solution by Jan van Delden (the Netherlands)

Using complex numbers, call the four corners (from A to D) $z_0 \dots z_3$. Then (with $z_4 = z_0$) the directions of the four sides are $z_{j+1} - z_j$, and the directions orthogonal to these sides are $(z_{j+1} - z_j) \cdot (-i)$; *i.e.*, we turn 90° counterclockwise to make the direction point outward from the quadrilateral. The centers of the four squares are therefore $C_j = 1/2 \cdot (z_{j+1} + z_j - i \cdot (z_{j+1} - z_j))$. The vector PR is then given by $C_2 - C_0 = 1/2 \cdot (z_3 + z_2 - i \cdot (z_3 - z_2) - (z_1 + z_0 - i \cdot (z_1 - z_0)))$, and similarly the vector QS is given by $C_3 - C_1 = 1/2 \cdot (z_0 + z_3 - i \cdot (z_0 - z_3) - (z_2 + z_1 - i \cdot (z_2 - z_1)))$. With a little bit of algebra one can show that we have

$$\begin{aligned} C_2 - C_0 &= 1/2 \cdot (a - ib) \\ C_3 - C_1 &= 1/2 \cdot (b + ia) && \text{with} \\ a &= z_2 - z_3 - z_1 - z_0 && \text{and} \\ b &= z_0 + z_3 - z_1 - z_2. \end{aligned}$$

We deduce $(C_3 - C_1) = i \cdot (C_2 - C_0)$. Hence QS is found from PR by rotating over $\pi/2$ anti-clockwise, and the lengths are equal since $|i| = 1$ and $|z1 \cdot z2| = |z1| \cdot |z2|$ for complex $z1$ and $z2$.

Also solved by Winston Teitler