Problem for 2015 April

Communicated by Dan Jurca

It is well known that if $p(t)$ is a polynomial of degree $n,\ 0 \le n$, with coefficients in a field F, then there exist no more than n roots of $p(t)$ in each extension field of F. According to Herstein (*Topics in Algebra*, page 180) there exist infinitely many roots of the polynomial t^2+1 in the (non-commutative) division ring Q, the quaternions. Find the set

$$\{q \in Q \mid q^2+1=0\}.$$

Solution by Dan Jurca

For each $q = w + xi + yj + zk \in Q$, where $w, x, y, z \in \mathbf{R},\ i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i$, and $ik = -j$, let $\overline{q} = w - xi - yj - zk$. Then one finds at once $q\overline{q} = \overline{q}q = w^2 + x^2 + y^2 + z^2$ and $q + \overline{q} = 2w$. Therefore $q^2 + q\overline{q} = 2wq$, so if $w = 0$ and $x^2 + y^2 + z^2 = 1$, then $q^2 + q\overline{q} = q^2 + 1 = 2 \cdot 0 \cdot q = 0$, and $q^2 + 1 = 0$. Therefore if (x, y, z) is a point on the unit sphere in $\mathbf{R}^3$ and $q = 0 + xi + yj + zk$, then $q^2 + 1 = 0$, so there exist uncountably many roots of the polynomial $t^2 + 1$ in Q.

To see that these are all the roots, suppose $q = w + xi + yj + zk$, where $w, x, y, z \in \mathbf{R}$; and $q^2 + 1 = 0$. Then from $q^2 + q\overline{q} = 2wq$ we find $-1 + q\overline{q} = 2wq$. If $w \neq 0$, then $q = (2w)^{-1}(-1 + q\overline{q}) = (2w)^{-1}(-1 + w^2 + x^2 + y^2 + z^2) \in \mathbf{R}$, contradicting $q^2 + 1 = 0$; therefore $w = 0$, and $q = xi + yj + zk$ where $q^2 + q\overline{q} = -1 + q\overline{q} = 0$, so $q\overline{q} = x^2 + y^2 + z^2 = 1$. Therefore

$$\{q \in Q \mid q^2 + 1 = 0\} = \{0 + xi + yj + zk \in Q \mid x, y, z \in \mathbf{R},\ \text{and}\ x^2 + y^2 + z^2 = 1\}.$$

Also solved by Massoud Malek and John M. Sayer