Problem for 2016 May

Proposed by Dan Jurca

According to problem 99 in *The Red Book of Mathematical Problems* by Hardy and Williams

$$\sum_{n=1}^{\infty} \frac{H_n}{n(n+1)} = \frac{\pi^2}{6},$$

where $H_n = 1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n$. The solution uses the following facts.

$$\lim_{n\to\infty} (H_n - \ln n) \text{ exists (and equals a certain number, } \gamma) \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Find $\displaystyle\sum_{n=1}^{\infty} \frac{H_n}{n(n+2)}$.

Solution by the proposer

For each positive integer N we let $S_N = \displaystyle\sum_{n=1}^{N} \frac{H_n}{n(n+2)}$.

Proposition. $1 \le N \;\Rightarrow\; 2S_N = 1 + \displaystyle\sum_{n=1}^{N} \frac{1}{n^2} + \frac{1}{(N+1)^2} - \frac{N+1}{(N+2)^2} - \frac{H_{N+1}}{N+1} - \frac{H_{N+2}}{N+2}$.

Proof.

$$2S_1 = 2\sum_{n=1}^{1} \frac{H_n}{n(n+2)} = \frac{2H_1}{1\cdot 3} = \frac{2}{3}, \quad \text{and}$$

$$\begin{aligned}
1 + \sum_{n=1}^{1} \frac{1}{n^2} + \frac{1}{(1+1)^2} - \frac{1+1}{(1+2)^2} - \frac{H_{1+1}}{1+1} - \frac{H_{1+2}}{1+2} &= 1 + 1 + \frac{1}{4} - \frac{2}{9} - \frac{3}{4} - \frac{11}{18} \\
&= \frac{36+36+9-8-27-22}{36} = \frac{81-57}{36} = \frac{24}{36} = \frac{2}{3},
\end{aligned}$$

so the asserted equality holds if $N = 1$.

If $2 \le N$ and

$$\begin{aligned}
2S_{N-1} &= 1 + \sum_{n=1}^{N-1} \frac{1}{n^2} + \frac{1}{N^2} - \frac{N}{(N+1)^2} - \frac{H_N}{N} - \frac{H_{N+1}}{N+1}, \quad \text{then} \\
2S_N &= 1 + \sum_{n=1}^{N-1} \frac{1}{n^2} + \frac{1}{N^2} - \frac{N}{(N+1)^2} - \frac{H_N}{N} - \frac{H_{N+1}}{N+1} + \frac{2H_N}{N(N+2)} \\
&= 1 + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{N}{(N+1)^2} - \frac{H_N}{N} - \frac{H_{N+1}}{N+1} + \frac{H_N}{N} - \frac{H_N}{N+2} \\
&= 1 + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{N}{(N+1)^2} - \frac{H_{N+1}}{N+1} - \frac{H_{N+2} - \dfrac{1}{N+1} - \dfrac{1}{N+2}}{N+2} \\
&= 1 + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{N}{(N+1)^2} + \frac{1}{(N+1)(N+2)} + \frac{1}{(N+2)^2} - \frac{H_{N+1}}{N+1} - \frac{H_{N+2}}{N+2} \\
&= 1 + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{N}{(N+1)^2} + \frac{1}{N+1} - \frac{1}{N+2} + \frac{1}{(N+2)^2} - \frac{H_{N+1}}{N+1} - \frac{H_{N+2}}{N+2} \\
&= 1 + \sum_{n=1}^{N} \frac{1}{n^2} - \frac{N}{(N+1)^2} + \frac{N+1}{(N+1)^2} - \frac{N+2}{(N+2)^2} + \frac{1}{(N+2)^2} - \frac{H_{N+1}}{N+1} - \frac{H_{N+2}}{N+2} \\
&= 1 + \sum_{n=1}^{N} \frac{1}{n^2} + \frac{1}{(N+1)^2} - \frac{N+1}{(N+2)^2} - \frac{H_{N+1}}{N+1} - \frac{H_{N+2}}{N+2},
\end{aligned}$$

and the proposition follows by induction on N.

Now since

$$\lim_{n\to\infty}(H_n - \ln n) = \gamma = 0.5772\ldots \quad \text{it follows that}$$

$$\lim_{n\to\infty}\frac{H_n}{n} = 0, \quad \text{and since also}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \quad \text{we find}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n(n+2)} &= \lim_{N\to\infty} S_N \\ &= \frac{1}{2} + \frac{\pi^2}{12}.\end{aligned}$$

Also solved by Massoud Malek, John M. Sayer, Benjamin Thomas, and Jan van Delden (the Netherlands)