Problem for 2016 October

Proposed by Matthew Hubbard

Suppose T is a triangle such that

- T is nondegenerate; *i.e.*, the area of T is strictly positive; and
- the length of each side of T is an integer; and
- the perimeter of T is odd.

Prove that the area of T is irrational.

Solution by Dan Jurca

Lemma. If n is an integer, then n^2 is congruent to 0, 1, or 4 (mod 8).

Proof.

If n is an integer, then (by the division theorem) there exist (unique) integers q and r such that $n = 4q+r$ and $0 \le r < 4$. Hence $n^2 = (4q + r)^2 = 16q^2 + 8qr + r^2 = 8(2q^2 + qr) + r^2$, and since $0 \le r < 4$, it follows that $r^2 = 0,\ 1,\ 4$, or 9, and the lemma follows.

Now suppose the lengths of the sides of T are $a,\ b$, and c, each a positive integer. Since $a + b + c$ is odd, it follows that either only one side is of odd length, or all three sides are of odd length.

So consider first for nonnegative integer x and positive integers y and z, that $a = 2x + 1,\ b = 2y$, and $c = 2z$. Using Heron's formula for the area A of T, $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$, we compute as follows.

$$\begin{aligned} s &= (1 + 2x + 2y + 2z)/2; \\ A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{(1 + 2x + 2y + 2z)(-1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z)}/4 \end{aligned}$$

It is well known and easy to prove that the square root of an integer N is rational if and only if N is a perfect square. Therefore A is rational if and only if the quantity, say Q_1, in the radical sign above is a perfect square. However, we find by a tedious computation (or using Mathematica) that

$$\begin{aligned} &(1 + 2x + 2y + 2z)(-1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z) \\ &= -1 - 8x - 24x^2 - 32x^3 - 16x^4 + 8y^2 + 32xy^2 + 32x^2y^2 - 16y^4 + 8z^2 + 32xz^2 + 32x^2z^2 + 32y^2z^2 - 16z^4 \\ &= 7 + 8(-1 - x - 3x^2 - 4x^3 - 2x^4 + y^2 + 4xy^2 + 4x^2y^2 - 2y^4 + z^2 + 4xz^2 + 4x^2z^2 + 4y^2z^2 - 2z^4), \end{aligned}$$

which is not congruent (mod 8) to 0, 1, or 4. Therefore Q_1 is not a perfect square, and A is irrational.

Next, suppose for nonnegative integers $x,\ y$, and z, that $a = 2x + 1,\ b = 2y + 1$, and $c = 2z + 1$. Using Heron's formula, we compute as follows.

$$\begin{aligned} s &= (3 + 2x + 2y + 2z)/2; \\ A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{(3 + 2x + 2y + 2z)(1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z)}/4 \\ &= \sqrt{Q_2}/4 \end{aligned}$$

Again, Q_2, the product of four integers, is an integer, and is A rational if and only if Q_2 is a perfect square. However, again tediously (or using Mathematica), we find

$$\begin{aligned} Q_2 &= (3 + 2x + 2y + 2z)(1 - 2x + 2y + 2z)(1 + 2x - 2y + 2z)(1 + 2x + 2y - 2z) \\ &= 3 + 8x - 8x^2 - 32x^3 - 16x^4 + 8y + 32xy + 32x^2y - 8y^2 + 32xy^2 + 32x^2y^2 - 32y^3 - 16y^4 + 8z + \\ &\quad 32xz + 32x^2z + 32yz + 32y^2z - 8z^2 + 32xz^2 + 32x^2z^2 + 32yz^2 + 32y^2z^2 - 32z^3 - 16z^4 \\ &= 3 + 8(x - x^2 - 4x^3 - 2x^4 + y + 4xy + 4x^2y - y^2 + 4xy^2 + 4x^2y^2 - 4y^3 - 2y^4 + z + \\ &\quad 4xz + 4x^2z + 4yz + 4y^2z - z^2 + 4xz^2 + 4x^2z^2 + 4yz^2 + 4y^2z^2 - 4z^3 - 2z^4), \end{aligned}$$

which is not congruent (mod 8) to 0, 1, or 4. Therefore Q_2 is not a perfect square, and A is irrational.