Problem for 2017 June

Proposed by Dan Jurca

1. Without evaluating either, determine which is greater: $\pi^{\sqrt{10}}$ or $\sqrt{10}^{\,\pi}$.
2. Show that e, the base of natural logarithms, is the only real number a such that $0 < x \;\Rightarrow\; x^a \le a^x$.

Solution by the proposer

The function $\varphi : (0,\infty) \to \mathbf{R}$ by $\varphi(x) = \ln x/x$ is differentiable, and since

$$0 < x \;\Rightarrow\; \varphi'(x) = \frac{1 - \ln x}{x^2}$$

it follows that $0 < x < e \;\Rightarrow\; 0 < \varphi'(x)$, so that φ strictly increases in $(0, e]$; $e < x \;\Rightarrow\; \varphi'(x) < 0$, so φ strictly decreases in $[e, \infty)$. Therefore $\varphi_{\max} = \varphi(e) = 1/e$, and this maximum value is attained at no other point.

1. Since $e \le a < b \;\Rightarrow\; \varphi(b) < \varphi(a)$, and

$$\begin{aligned} \pi &< \frac{22}{7}, \quad \text{it follows that} \\ \pi^2 &< \frac{484}{49} \\ &< \frac{490}{49} \\ &= 10, \quad \text{so that} \\ \pi &< \sqrt{10}, \end{aligned}$$

and since $e < 3 < \pi < \sqrt{10}$, we have $\varphi(\sqrt{10}) < \varphi(\pi)$. Therefore

$$\begin{aligned} \frac{\ln\sqrt{10}}{\sqrt{10}} &< \frac{\ln \pi}{\pi}, \quad \text{so} \\ \pi \ln\sqrt{10} &< \sqrt{10}\ln \pi, \quad \text{whence} \\ \sqrt{10}^{\,\pi} &< \pi^{\sqrt{10}}. \end{aligned}$$

2. First, since $\varphi_{\max} = \varphi(e) = 1/e$, we have $0 < x \;\Rightarrow\; \varphi(x) \le 1/e$; *i.e.*, $\ln x/x \le 1/e$, so that

$$e \ln x \le x = x \ln e;$$

hence $0 < x \;\Rightarrow\; x^e \le e^x$.

Now suppose $0 < a$, and $0 < x \;\Rightarrow\; x^a \le a^x$. Then (since $0 < e$) by hypothesis $e^a \le a^e$; but then by what we have just shown, we find $e^a \le a^e \le e^a$. Therefore $e^a = a^e$. But then $a = e \ln a$, whence $\ln a/a = 1/e$, so that $\varphi(a) = \varphi_{\max}$, which is attained only at e; therefore $a = e$.

We may summarize the above as follows.

$$\begin{aligned} 0 < a < x \le e \Rightarrow\; & \varphi(a) < \varphi(x) \;\Rightarrow\; a^x < x^a \\ e \le x < a \Rightarrow\; & \varphi(a) < \varphi(x) \;\Rightarrow\; a^x < x^a; \\ 0 < x \Rightarrow\; & \varphi(x) \le \varphi(e) \;\Rightarrow\; x^e \le e^x \end{aligned}$$

Also solved by John M. Sayer