Problem for 2017 July

Communicated by Dan Jurca

$f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 120x^7 \sin^2(x^{1,000})e^{x^2}$; evaluate the 2,017-th derivative of f at 0, $f^{(2017)}(0)$.

Solution by Dan Jurca

Since f is analytic in $\mathbf{R}$,

$$x \in \mathbf{R} \;\Rightarrow\; f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n,$$

so that $f^{(2017)}(0)$ equals 2,017! times the coefficient of $x^{2,017}$ in the Maclaurin series for f. Then since

$$\begin{aligned} f(x) &= 120x^7 \cdot \left(\sum_{i=0}^{\infty} (-1)^i \frac{(x^{1,000})^{2i+1}}{(2i+1)!} \right)^2 \cdot \sum_{j=0}^{\infty} \frac{(x^2)^j}{j!} \\ &= 120x^7 \cdot \left(x^{1,000} - \frac{x^{3,000}}{3!} + - \cdots \right)^2 \cdot \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots \right) \\ &= 120x^7 \cdot \left(x^{2,000} - + \cdots \right) \cdot \left(1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \cdots + \frac{x^{10}}{5!} + \cdots \right) \end{aligned}$$

the coefficient of x^{2017} in that series equals

$$120 \cdot 1 \cdot \frac{1}{5!} = 1.$$

Therefore $f^{(2017)}(0) = 2{,}017!$.

Also solved by Jan van Delden (the Netherlands), John M. Sayer, and Winston Teitler