Problem for 2017 August

Proposed by Dan Jurca

For $n = 1, 2, 3, \ldots$ let

$$a_n = 1 \times 4 \times 7 \times 10 \times \cdots \times (3n-2) \quad \text{and}$$
$$b_n = 2 \times 5 \times 8 \times 11 \times \cdots \times (3n-1).$$

Does the sequence $(b_n/a_n)_{n=1}^{\infty}$ converge?

Solution 1 by the proposer

No; the sequence increases and diverges to ∞. For $b_1/a_1 = 2$, and

$$2 \le n \;\Rightarrow\; \frac{b_n}{a_n} = \frac{b_{n-1}}{a_{n-1}} \cdot \frac{3n-1}{3n-2} = \frac{b_{n-1}}{a_{n-1}} \cdot \left(1 + \frac{1}{3n-2}\right),$$

so the sequence strictly increases, and we have the following

Proposition 1. $1 \le n \;\Rightarrow\; 2 \cdot \sum_{i=1}^{n} \frac{1}{3i-2} \le \frac{b_n}{a_n}$; (equality iff $n \le 2$).

Proof.

$$2 \cdot \sum_{i=1}^{1} \frac{1}{3i-2} = 2 \cdot \frac{1}{1} = \frac{2}{1} = \frac{b_1}{a_1},$$

so the assertion holds if $n = 1$. If $2 \le n$ and

$$2 \cdot \sum_{i=1}^{n-1} \frac{1}{3i-2} \le \frac{b_{n-1}}{a_{n-1}}, \qquad \text{then}$$

$$\begin{aligned}
2 \cdot \sum_{i=1}^{n} \frac{1}{3i-2} &= 2 \cdot \left(\sum_{i=1}^{n-1} \frac{1}{3i-2} + \frac{1}{3n-2}\right) \\
&= 2 \cdot \left(\sum_{i=1}^{n-1} \frac{1}{3i-2} + 1 \cdot \frac{1}{3n-2}\right) \\
&\le 2 \cdot \left(\sum_{i=1}^{n-1} \frac{1}{3i-2} + \sum_{i=1}^{n-1} \frac{1}{3i-2} \cdot \frac{1}{3n-2}\right) \\
&= 2 \cdot \left(\sum_{i=1}^{n-1} \frac{1}{3i-2} \left(1 + \frac{1}{3n-2}\right)\right) \\
&= 2 \cdot \left(\sum_{i=1}^{n-1} \frac{1}{3i-2} \cdot \frac{3n-1}{3n-2}\right) \\
&= \left(2 \cdot \sum_{i=1}^{n-1} \frac{1}{3i-2}\right) \cdot \frac{3n-1}{3n-2} \\
&\le \frac{b_{n-1}}{a_{n-1}} \cdot \frac{3n-1}{3n-2} \\
&= \frac{b_n}{a_n},
\end{aligned}$$

and the proposition follows by induction on n.

Since (by the "limit comparison test" and the fact that the harmonic series diverges) $\sum_{i=1}^{\infty} \frac{1}{3i-2} = \infty$, it follows that $(b_n/a_n)_{n=1}^{\infty} \to \infty$.

Solution 2 by the proposer

Proposition 2. $1 \le n \;\Rightarrow\; \left(\dfrac{3n+2}{2}\right)^{1/3} < \dfrac{b_n}{a_n}$.

Proof.

Since $5 < 16$, we have $5/2 < 8$, whence $(5/2)^{1/3} < 2$, so $((3\cdot 1+2)/2)^{1/3} < 2/1 = b_1/a_1$, and the assertion holds for the case $n = 1$. Next,

$$\begin{aligned}
n \in \mathbf{R} \;\Rightarrow\; 0 &< 54(n-5/9)^2 + 1/3 \\
&= 54(n^2 - 10/9\cdot n + 25/81) + 1/3 \\
&= 54n^2 - 60n + 50/3 + 1/3 \\
&= 54n^2 - 60n + 17; \quad \text{therefore} \\
81n^4 - 108n^3 + 48n - 16 &< 81n^4 - 108n^3 + 54n^2 - 12n + 1, \quad \text{so} \\
(3n-2)^3(3n+2) &< (3n-1)^4.
\end{aligned}$$

$$\begin{aligned}
\text{Hence} \quad 1 \le n \;\Rightarrow\; \frac{3n+2}{3n-1} &< \left(\frac{3n-1}{3n-2}\right)^3, \quad \text{so} \\
\frac{3n+2}{2} &< \frac{3n-1}{2}\cdot\left(\frac{3n-1}{3n-2}\right)^3, \quad \text{and} \\
1 \le n \;\Rightarrow\; \left(\frac{3n+2}{2}\right)^{1/3} &< \left(\frac{3n-1}{2}\right)^{1/3}\cdot\frac{3n-1}{3n-2}.
\end{aligned}$$

Now suppose that $2 \le n$ and (inductively)

$$\begin{aligned}
\left(\frac{3n-1}{2}\right)^{1/3} &< \frac{b_{n-1}}{a_{n-1}}; \quad \text{then} \\
\left(\frac{3n-1}{2}\right)^{1/3}\cdot\frac{3n-1}{3n-2} &< \frac{b_{n-1}}{a_{n-1}}\cdot\frac{3n-1}{3n-2} \\
&= \frac{b_n}{a_n}; \quad \text{but then} \\
\left(\frac{3n+2}{2}\right)^{1/3} &< \frac{b_n}{a_n},
\end{aligned}$$

and the proposition follows by induction on n.

Thus, again, $(b_n/a_n)_{n=1}^{\infty} \to \infty$.

Solution 3 by the proposer

The fact that $b_n/a_n \sim L\sqrt[3]{n}$ for some L, $1 < L < 2$, follows from

Proposition 3. There exists $L \in \mathbf{R}$, $1 < L < 2$, such that

$$\lim_{n\to\infty} \frac{b_n/a_n}{\sqrt[3]{n}} = L.$$

Proof.

If for $1 \le n$ we let $q_n = b_n/a_n$, then we show $1 \le n \Rightarrow q_n \le 2\sqrt[3]{n}$ by induction on n, as follows. Since $q_1 = 2/1 = 2 = 2\sqrt[3]{1}$, this certainly holds if $n = 1$. Next

$$\begin{aligned} 1 < n \;\Rightarrow\; 1 < 2n, &\quad \text{so} \\ -10n + 1 < -8n, &\quad \text{so} \\ 27n^4 - 54n^3 + 36n^2 - 10n + 1 < 27n^4 - 54n^3 + 36n^2 - 8n, &\quad \text{so} \\ (n-1)(3n-1)^3 < n(3n-2)^3, &\quad \text{so} \\ \frac{3n-1}{3n-2} < \left(\frac{n}{n-1}\right)^{1/3}. & \end{aligned}$$

Hence, if $2 \le n$ and $q_{n-1} \le 2\sqrt[3]{n-1}$, then, since $q_n = q_{n-1} \cdot (3n-1)/(3n-2)$, it follows that $q_n \le 2\sqrt[3]{n}$, and this inequality is strict if $2 \le n$. From this same inequality above we deduce

$$2 \le n \;\Rightarrow\; \frac{q_n}{\sqrt[3]{n}} = \frac{(3n-1)/(3n-2) \cdot q_{n-1}}{\sqrt[3]{n}} < \frac{q_{n-1}}{\sqrt[3]{n-1}},$$

so that the sequence $(q_n/\sqrt[3]{n})_{n=1}^{\infty}$ strictly decreases. We next show this sequence is bounded below by a number greater than 1. (This follows also from proposition 2: $\sqrt[3]{3/2} < q_n/\sqrt[3]{n}$.)

Since

$$\begin{aligned} 1 \le n \;\Rightarrow\; q_n &= \frac{2}{1} \times \frac{5}{4} \times \frac{8}{7} \times \cdots \times \frac{3n-1}{3n-2} \\ &= (1 + 1/1) \times (1 + 1/4) \times (1 + 1/7) \times \cdots \times (1 + 1/(3n-2)), \quad \text{we have} \\ \ln q_n &= \sum_{i=1}^{n} \ln\left(1 + \frac{1}{3i-2}\right). \end{aligned}$$

Using the ("well-known and easily proved") inequality $-1 < x \Rightarrow x/(1+x) \le \ln(1+x)$, we find

$$1 \le i \;\Rightarrow\; \frac{1/(3i-2)}{1 + 1/(3i-2)} = \frac{1}{3i-1} \le \ln\left(1 + \frac{1}{3i-2}\right).$$

Therefore

$$1 \le n \;\Rightarrow\; \frac{1}{3}H_n = \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i} < \frac{1}{3}\sum_{i=1}^{n} \frac{1}{i - 1/3} = \sum_{i=1}^{n} \frac{1}{3i-1} \le \sum_{i=1}^{n} \ln\left(1 + \frac{1}{3i-2}\right) = \ln q_n.$$

Hence $1 \le n \Rightarrow H_n < 3\ln q_n$, so $H_n - \ln n < 3\ln q_n - \ln n$. But the sequence $(H_n - \ln n)_{n=1}^{\infty}$ strictly decreases, and converges to (the "Euler-Mascheroni constant") $\gamma = 0.5772156649\ldots$. Hence

$$1 \le n \;\Rightarrow\; \gamma < H_n - \ln n < 3\ln q_n - \ln n = 3(\ln q_n - (\ln n)/3) = 3(\ln q_n - \ln\sqrt[3]{n}) = 3\ln(q_n/\sqrt[3]{n}).$$

Therefore $1 \le n \Rightarrow \gamma/3 < \ln(q_n/\sqrt[3]{n})$, so $e^{\gamma/3} < q_n/\sqrt[3]{n}$, and the decreasing and bounded below sequence $(q_n/\sqrt[3]{n})_{n=1}^{\infty}$ converges to a number L, where $1.2121616\ldots = e^{\gamma/3} \le L$, proving the proposition.

Remark. Thus $1 \le n \Rightarrow L\sqrt[3]{n} < b_n/a_n \le 2\sqrt[3]{n}$. Numerical computation suggests, and the proposer conjectures, that in fact $L \approx 1.9783642596$; the proposer does not recognize this number.

Also solved by John Sayer and Jan van Delden (the Netherlands)