Subject: Algebra/ Calculus
Created by: Sabrina Voelker
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Horizontal and Vertical Asymptotes

Finding Asymptotes

- **Horizontal Asymptotes**: Horizontal asymptotes are horizontal lines on your graph which your function reaches for as **x** goes to infinity. You find your H.A. by taking the limit of the function as **x** goes to infinity. (See "Limits to Infinity" for elaboration)

Example A

$$f(x) = \frac{2x^2+4x-16}{x^2-7x+12}$$

$$\lim_{x\to\infty} \frac{2x^2+4x-16}{x^2-7x+12}$$

$$\lim_{x\to\infty} \frac{2x^2+4x-16}{x^2-7x+12} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{4x}{x^2}-\frac{16}{x^2}}{\frac{x^2}{x^2}-\frac{7x}{x^2}+\frac{12}{x^2}}$$

$$\lim_{x\to\infty} \frac{2+\frac{4}{x}-\frac{16}{x^2}}{1-\frac{7}{x}+\frac{12}{x^2}}$$

$$= \frac{2+0-0}{1-0+0} = \frac{2}{1} = 2$$

Which means we have H.A. at:
$y = f(x) = 2$

Example B (*A Trickier Problem*)

$$f(x) = \frac{\sqrt{x+4}}{x^2-x-20}$$

$$\lim_{x\to\infty} \frac{\sqrt{x+4}}{x^2-x-20}$$

$$\lim_{x\to\infty} \frac{\sqrt{x+4}}{x^2-x-20} \cdot \frac{\frac{1}{\sqrt{x^4}}}{\frac{1}{x^2}}$$

$$\lim_{x\to\infty} \frac{\sqrt{\frac{x}{x^4}+\frac{4}{x^4}}}{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{20}{x^2}}$$

$$\lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^3}+\frac{4}{x^4}}}{1-\frac{1}{x}-\frac{20}{x^2}}$$

$$= \frac{\sqrt{0+0}}{1-0-0} = \frac{0}{1} = 0$$

Which means we have H.A. at:
$y = f(x) = 0$

- **Vertical Asymptotes**: Vertical asymptotes are vertical lines on your graph which a function can never touch. They occur because, at those values, the function cannot exist. Examples of this are values of **x** which make the denominator of a fraction zero or the value underneath a square root negative.

Example A

$$f(x) = \frac{2x^2+4x-16}{x^2-7x+12}$$

Example B

$$f(x) = \frac{\sqrt{x+4}}{x^2-x-20}$$

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Horizontal and Vertical Asymptotes

$$f(x) = \frac{(2x+8)(x-2)}{(x-4)(x-3)}$$

Denominator Cannot Be Zero

$(x-4)(x-3) \neq 0$

$(x-4) \neq 0 \qquad (x-3) \neq 0$

$x \neq 4 \qquad x \neq 3$

Which means we have V.A. at:
$x = 3, 4$

Note: In cases where you can cancel a set of parentheses, the factor eliminated becomes a hole rather than a V.A.

In Example: $$f(x) = \frac{(x+5)}{(x+5)(x-7)}$$
$$= \frac{1}{(x-7)} \text{ where } x \neq -5$$

There is a V.A. at $x = 7$ in this case and a hole at $x = -5$

$$f(x) = \frac{\sqrt{x+4}}{(x+4)(x-5)}$$

Denominator Cannot Be Zero

$(x+4)(x-5) \neq 0$

$(x+4) \neq 0 \qquad (x-5) \neq 0$

$x \neq -4 \qquad x \neq 5$

Value Under a Square Root Cannot be Negative

$\sqrt{x+4}$

$x + 4 \geq 0$

$x \geq -4$

However, because $x \neq -4$, these properties stack. So, $x > -4$
Which means we have V.A. at:
$x = -4, 5$

Properties at those Asymptotes

- **Does it cross the Horizontal Asymptote?** Although the H.A. is a line your function ultimately reaches for, that doesn't mean it never touches the line. To see if the function crosses the H.A., simply set the equation equal to that number and solve to see what values of **x** (if any) allow it to touch that line.

Example A

$$f(x) = \frac{2x^2+4x-16}{x^2-7x+12}$$

$$2 = \frac{2x^2+4x-16}{x^2-7x+12}$$

Example B

$$f(x) = \frac{\sqrt{x+4}}{x^2-x-20}$$

$$0 = \frac{\sqrt{x+4}}{x^2-x-20}$$

Horizontal and Vertical Asymptotes

$2(x^2 - 7x + 12) = 2x^2 + 4x - 16$

$2x^2 - 14x + 24 = 2x^2 + 4x - 16$

$-14x + 24 = 4x - 16$

$24 = 18x - 16$

$40 = 18x$

$x = \frac{40}{18} = \frac{20}{9}$

so it does cross the H.A at: $x = \frac{20}{9}$

$0 = \sqrt{x+4}$

$0^2 = (\sqrt{x+4})^2$

$0 = x + 4$

$x = -4$ is our solution , but $x \neq -4$ so it does not cross the H.A.

- **Behavior at Vertical Asymptotes:** When a function reaches a Vertical Asymptote, it goes toward positive or negative infinity. To see which occurs on either side of the line, take the limit by plugging in values just smaller than and just greater than those values at which the V.A. occurs. Here, we only care about whether the values of these factors are positive or negative. The outcome then tells us whether the function heads toward positive or negative infinity at that particular limit.

Example A

We have V.A. at:

$x = 3, 4$

$$f(x) = \frac{2x^2 + 4x - 16}{x^2 - 7x + 12}$$

$$f(x) = \frac{(2x+8)(x-2)}{(x-4)(x-3)}$$

$$\lim_{x \to 3^-} \frac{(2x+8)(x-2)}{(x-4)(x-3)}$$

$$= \frac{(2(3^-)+8)((3^-)-2)}{((3^-)-4)((3^-)-3)}$$

Example B

We have V.A. at:

$x = -4, 5$

$$f(x) = \frac{\sqrt{x+4}}{x^2 - x - 20}$$

$$f(x) = \frac{\sqrt{x+4}}{(x+4)(x-5)}$$

Note: We do not take the limit as it approaches to the left because those values are less than -4, which makes the function incomputable.

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Horizontal and Vertical Asymptotes

$$= \frac{(2(2.9)+8)((2.9)-2)}{((2.9)-4)((2.9)-3)}$$

$$= \frac{(+)(+)}{(-)(-)} = \frac{(+)}{(+)} = (+)$$

$$\Rightarrow \lim_{x \to 3^-} f(x) = +\infty$$

Find the left and right limit of all the values of x which cause V.A. to find the behavior of $f(x)$ as it approaches them

$$\lim_{x \to -4^+} \frac{\sqrt{x+4}}{(x+4)(x-5)}$$

$$= \frac{\sqrt{-4^+ + 4}}{(-4^+ + 4)(-4^+ - 5)}$$

$$= \frac{\sqrt{(-3.9)+4}}{((-3.9)+4)((-3.9)-5)}$$

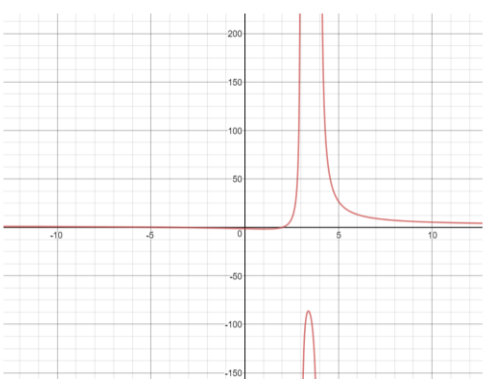
$$= \frac{(+)}{(+)(-)} = \frac{(+)}{(-)} = (-)$$

$$\Rightarrow \lim_{x \to -4^+} f(x) = -\infty$$

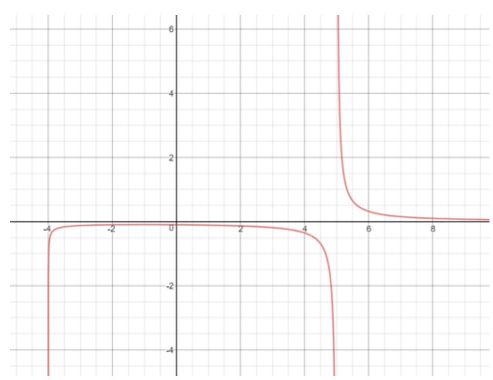
Graphing

- Draw Vertical lines to represent your V.A.
- Draw a Horizontal line to represent your H.A.
- Plot where x crosses the H.A. (if it does)
- Use the behavior of the function near the V.A. to determine if it goes toward positive or negative infinity as it approaches those values

Example A



Example B



Reference: *Stewart James, Algebra and Trigonometry, 4th Ed.*

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