

# L'Hôpital's Rule

## Indeterminate forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, (0)(\pm\infty), 1^\infty, 0^0, \infty^0, (\infty - \infty)$$

### ➤ L'Hôpital's Rule:

- When to use L'Hôpital's Rule:

$$\text{If; } \lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}$$

$$\text{Then; } \lim_{x \rightarrow a} \left[ \frac{f(x)}{g(x)} \right] = \lim_{x \rightarrow a} \left[ \frac{f'(x)}{g'(x)} \right]$$

Notice from the rule that we can only apply L'Hôpital's Rule when we have the limit of a **quotient** that yields an indeterminate form on direct substitution. Hence to apply L'Hôpital's Rule we must first have a **quotient i.e.**  $\left[ \frac{a}{b} \right]$

- When we should **not** use L'Hôpital's Rule:

✓ When we are not taking the limit of a quotient.

✓ When we do not have the indeterminate forms  $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$ .

Before using L'Hôpital's Rule, we must be sure that we have obtained one of the above indeterminate forms from direct substitution. If this condition is not met, application of L'Hôpital's Rule will most likely result in a wrong answer.

Try  $\lim_{x \rightarrow 0} \frac{x^2 - 3}{x^2 + 2}$ . First use direct substitution ( $= \frac{-3}{2}$ ). Using L'Hôpital's Rule we get "1/0 (Undefined)  $\frac{-3}{2}$ " as the answer. The answer from L'Hôpital's Rule is incorrect in this case because we do not have the indeterminate forms;  $\frac{0}{0}$  or  $\frac{\pm\infty}{\pm\infty}$ .

✓ When it further complicates the problem.

$$\lim_{x \rightarrow \infty} \left[ \frac{x}{(x^2 + 1)^{\frac{1}{2}}} \right] \text{ Indeterminate form, } \frac{\infty}{\infty} \dots$$

$$\text{(By L'Hôpital's Rule)} = \lim_{x \rightarrow \infty} \left[ \frac{\frac{d(x)}{dx}}{\frac{d((x^2 + 1)^{\frac{1}{2}})}{dx}} \right] = \lim_{x \rightarrow \infty} \left[ \frac{1}{\frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}(2x)} \right] = \lim_{x \rightarrow \infty} \left[ \frac{(x^2 + 1)^{\frac{1}{2}}}{x} \right] \text{ Indeterminate}$$

form,  $\frac{\infty}{\infty}$  ... apply L'Hôpital's Rule again,



$$= \lim_{x \rightarrow \infty} \left[ \frac{\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)}{1} \right] = \lim_{x \rightarrow \infty} \left[ \frac{x}{(x^2+1)^{\frac{1}{2}}} \right] \text{ Indeterminate form, } \frac{\infty}{\infty} \dots$$

*NB:* This result is the same as the original problem. This shows that if we keep applying L'Hôpital's Rule we will keep going in circles indefinitely. Therefore L'Hôpital's Rule does not work here. Note that this does not mean that the limit does not exist, only that we will have to use some other method to find this limit.

### Using L'Hôpital's Rule:

1.  $\lim_{x \rightarrow \infty} \left[ \frac{2x^2}{3x^2-1} \right]$  Indeterminate form,  $\frac{\infty}{\infty}$  ...

(By L'Hôpital's Rule)  $= \lim_{x \rightarrow \infty} \left[ \frac{\frac{d(2x^2)}{dx}}{\frac{d(3x^2-1)}{dx}} \right] = \lim_{x \rightarrow \infty} \left[ \frac{4x}{6x} \right] = \lim_{x \rightarrow \infty} \left[ \frac{4}{6} \right] = \frac{4}{6}$

2.  $\lim_{x \rightarrow 0} \left[ \frac{e^x - e^{-x}}{\sin(x)} \right]$  Indeterminate form,  $\frac{0}{0}$  ...

(By L'Hôpital's Rule)  $= \lim_{x \rightarrow 0} \left[ \frac{\frac{d(e^x - e^{-x})}{dx}}{\frac{d(\sin(x))}{dx}} \right] = \lim_{x \rightarrow 0} \left[ \frac{e^x + e^{-x}}{\cos(x)} \right] = \frac{e^0 + e^0}{\cos(0)} = 2$

❖ Sometimes when solving limit problems that end in indeterminate form we need to modify the problem a little bit to get quotients that can be solved using L'Hôpital's rule.

3.  $\lim_{x \rightarrow 0^+} [x \ln(x)]$  Indeterminate form  $(0)(-\infty)$  ...

$\lim_{x \rightarrow 0^+} [x \ln(x)] = \lim_{x \rightarrow 0^+} \left[ \frac{\ln(x)}{x^{-1}} \right]$  Indeterminate form,  $\frac{-\infty}{\infty}$

(By L'Hôpital's Rule)  $= \lim_{x \rightarrow 0^+} \left[ \frac{\frac{d[\ln(x)]}{dx}}{\frac{d(x^{-1})}{dx}} \right] = \lim_{x \rightarrow 0^+} \left[ \frac{1/x}{-x^{-2}} \right] = \lim_{x \rightarrow 0^+} [-x] = 0$

4.  $\lim_{x \rightarrow \infty} -[x e^x]$  Indeterminate form,  $(-\infty)(0)$ ! ...

$\lim_{x \rightarrow \infty} -[x e^x] = \lim_{x \rightarrow \infty} \left[ \frac{x}{e^{-x}} \right]$  Indeterminate form,  $\frac{-\infty}{\infty}$

(By L'Hôpital's Rule)  $= \lim_{x \rightarrow \infty} \left[ \frac{\frac{d(x)}{dx}}{\frac{d(e^{-x})}{dx}} \right] = \lim_{x \rightarrow \infty} \left[ \frac{1}{-e^{-x}} \right] = \lim_{x \rightarrow \infty} [-e^x] = 0$

\*notice that pushing  $x$  to the denominator instead of  $e^x$  in this case further compounds the problem (Try it out yourself). Sometimes when solving problems we have to try different routes to get to the solution.



5.  $\lim_{x \rightarrow 0^+} [x^x]$  Indeterminate form  $0^0 \dots$

To solve this problem we have to make work with a different problem that will help us get to the solution of this one.

$$\text{Let } y = x^x$$

$$\rightarrow \ln(y) = \ln(x^x)$$

$$\rightarrow \ln(y) = x \ln(x)$$

$$\rightarrow \lim_{x \rightarrow 0^+} [\ln(y)] = \lim_{x \rightarrow 0^+} [x \ln(x)] \dots$$

$$\text{Recall from \#4 } \lim_{x \rightarrow 0^+} [x \ln(x)] = 0$$

$$\rightarrow \lim_{x \rightarrow 0^+} [\ln(x)] = 0$$

$$\rightarrow \lim_{x \rightarrow 0^+} [e^{\ln(y)}] = e^0$$

$$\rightarrow \lim_{x \rightarrow 0^+} [y] = 1$$

$$\text{But } y = x^x$$

$$\rightarrow \lim_{x \rightarrow 0^+} [x^x] = 1$$

6.  $\lim_{x \rightarrow \infty} \left[ (1 + e^{2x})^{\frac{1}{x}} \right]$  Indeterminate form  $\infty^0$

Just like above ...

$$\text{Let } y = (1 + e^{2x})^{\frac{1}{x}}$$

$$\rightarrow \ln(y) = \ln(1 + e^{2x})^{\frac{1}{x}}$$

$$\rightarrow \ln(y) = \frac{1}{x} \cdot \ln(1 + e^{2x}) = \frac{\ln(1 + e^{2x})}{x}$$

$$\rightarrow \lim_{x \rightarrow \infty} [\ln(y)] = \lim_{x \rightarrow \infty} \left[ \frac{\ln(1 + e^{2x})}{x} \right] \text{ Indeterminate form } \frac{\infty}{\infty}$$

$$\text{(By L'H\^opital's Rule)} = \lim_{x \rightarrow \infty} \left[ \frac{\frac{2e^{2x}}{1 + e^{2x}}}{1} \right] = \lim_{x \rightarrow \infty} \left[ \frac{2e^{2x}}{1 + e^{2x}} \right] \text{ I.F. } \frac{\infty}{\infty}$$

$$\text{(By L'H\^opital's Rule)} = \lim_{x \rightarrow \infty} \left[ \frac{4e^{2x}}{2e^{2x}} \right] = \lim_{x \rightarrow \infty} [2] = 2$$

$$\rightarrow \lim_{x \rightarrow \infty} [\ln(y)] = 2$$

$$\rightarrow \lim_{x \rightarrow \infty} [e^{\ln(y)}] = e^2$$

$$\rightarrow \lim_{x \rightarrow \infty} [y] = e^2$$

$$\text{But } y = (1 + e^{2x})^{\frac{1}{x}}$$

$$\rightarrow \lim_{x \rightarrow \infty} \left[ (1 + e^{2x})^{\frac{1}{x}} \right] = e^2$$

