Differential Equations Reference Sheet

Derivatives

1. $\dfrac{d}{dx}\left[f(u)\cdot g(u)\right] = \left[f'(u)\cdot g(u) + f(u)\cdot g'(u)\right]u'$
2. $\dfrac{d}{dx}\left[\dfrac{f(u)}{g(u)}\right] = \left[\dfrac{f'(u)\cdot g(u) - f(u)\cdot g'(u)}{\left[g(u)\right]^2}\right]u'$
3. $\dfrac{d}{dx}\left[u^n\right] = \left[nu^{n-1}\right]u'$
4. $\dfrac{d}{dx}\left[\ln u\right] = \left[\dfrac{1}{u}\right]u'$
5. $\dfrac{d}{dx}\left[\log_a u\right] = \left[\dfrac{1}{u\ln a}\right]u'$
6. $\dfrac{d}{dx}\left[e^u\right] = \left[e^u\right]u'$
7. $\dfrac{d}{dx}\left[a^u\right] = \left[a^u\ln a\right]u'$
8. $\dfrac{d}{dx}\left[\sin u\right] = \left[\cos u\right]u'$
9. $\dfrac{d}{dx}\left[\cos u\right] = \left[-\sin u\right]u'$
10. $\dfrac{d}{dx}\left[\tan u\right] = \left[\sec^2 u\right]u'$
11. $\dfrac{d}{dx}\left[\cot u\right] = \left[-\csc^2 u\right]u'$
12. $\dfrac{d}{dx}\left[\sec u\right] = \left[\sec u\tan u\right]u'$
13. $\dfrac{d}{dx}\left[\csc u\right] = \left[-\csc u\cot u\right]u'$
14. $\dfrac{d}{dx}\left[\arcsin u\right] = \left[\dfrac{1}{\sqrt{1-u^2}}\right]u'$
15. $\dfrac{d}{dx}\left[\arccos u\right] = \left[\dfrac{-1}{\sqrt{1-u^2}}\right]u'$
16. $\dfrac{d}{dx}\left[\arctan u\right] = \left[\dfrac{1}{1+u^2}\right]u'$
17. $\dfrac{d}{dx}\left[\operatorname{arccot} u\right] = \left[\dfrac{-1}{1+u^2}\right]u'$
18. $\dfrac{d}{dx}\left[\operatorname{arcsec} u\right] = \left[\dfrac{1}{|u|\sqrt{u^2-1}}\right]u'$
19. $\dfrac{d}{dx}\left[\operatorname{arccsc} u\right] = \left[\dfrac{-1}{|u|\sqrt{u^2-1}}\right]u'$

Integrals

1. $\displaystyle\int u^a\,du = \frac{u^{a+1}}{a+1} + C, \quad a \neq -1$
2. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$
3. $\displaystyle\int e^u\,du = e^u + C$
4. $\displaystyle\int a^u\,du = \frac{1}{\ln a}a^u + C$
5. $\displaystyle\int \sin u\,du = -\cos u + C$
6. $\displaystyle\int \cos u\,du = \sin u + C$
7. $\displaystyle\int \sec^2 u\,du = \tan u + C$
8. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
9. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$
10. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
11. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
12. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$
13. $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$
14. $\displaystyle\int \frac{1}{\sqrt{u^2\pm a^2}}\,du = \ln\left[u + \sqrt{u^2\pm a^2}\right] + C$
15. $\displaystyle\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$
16. $\displaystyle\int \frac{1}{u\sqrt{a^2\pm u^2}}\,du = \frac{-1}{a}\ln\frac{a+\sqrt{a^2\pm u^2}}{|u|} + C$
17. $\displaystyle\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\frac{u}{a} + C$
18. $\displaystyle\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C = \frac{1}{a}\operatorname{arctanh}\frac{u}{a} + C$
19. $\displaystyle\int \sqrt{a^2-u^2}\,du = \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2\arctan\left[\frac{u}{\sqrt{a^2-u^2}}\right]\right) + C$
20. $\displaystyle\int \sqrt{u^2\pm a^2}\,du = \frac{1}{2}\left(u\sqrt{u^2\pm a^2} \pm a^2\ln\left[u + \sqrt{u^2\pm a^2}\right]\right) + C$
21. $\displaystyle\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}\left[a\cos bu + b\sin bu\right] + C$
22. $\displaystyle\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}\left[a\sin bu - b\cos bu\right] + C$

Laplace Transforms

1. $\mathcal{L}\{f(t)\} = F(s)$
2. $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
3. $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$
4. $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$
5. $\mathcal{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
6. $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s}$
7. $\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$
8. $\mathcal{L}\{u_a(t)f(t-a)\} = e^{-as}F(s)$
9. $\mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)G(s)$
10. $\mathcal{L}\{tf(t)\} = -F'(s)$
11. $\mathcal{L}\{t^nf(t)\} = (-1)^nF^{(n)}(s)$
12. $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_s^\infty F(\sigma)\,d\sigma$
13. $\mathcal{L}\{f(t), \text{ period } p\} = \dfrac{1}{1-e^{-ps}}\displaystyle\int_0^p e^{-st}f(t)\,dt$
14. $\mathcal{L}\{1\} = \dfrac{1}{s}$
15. $\mathcal{L}\{t\} = \dfrac{1}{s^2}$
16. $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$
17. $\mathcal{L}\left\{\dfrac{1}{\sqrt{\pi t}}\right\} = \dfrac{1}{\sqrt{s}}$
18. $\mathcal{L}\{t^a\} = \dfrac{\Gamma(a+1)}{s^{a+1}}$
19. $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{s-a}$
20. $\mathcal{L}\left\{t^ne^{at}\right\} = \dfrac{n!}{(s-a)^{n+1}}$
21. $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$
22. $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$
23. $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$
24. $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$
25. $\mathcal{L}\left\{e^{at}\cos kt\right\} = \dfrac{s-a}{(s-a)^2+k^2}$
26. $\mathcal{L}\left\{e^{at}\sin kt\right\} = \dfrac{k}{(s-a)^2+k^2}$
27. $\mathcal{L}\left\{\dfrac{1}{2k^3}(\sin kt - kt\cos kt)\right\} = \dfrac{1}{(s^2+k^2)^2}$
28. $\mathcal{L}\left\{\dfrac{t}{2k}\sin kt\right\} = \dfrac{s}{(s^2+k^2)^2}$
29. $\mathcal{L}\left\{\dfrac{1}{2k}(\sin kt + kt\cos kt)\right\} = \dfrac{s^2}{(s^2+k^2)^2}$
30. $\mathcal{L}\{u_a(t)\} = \dfrac{e^{-as}}{s}$
31. $\mathcal{L}\{\delta_a(t)\} = e^{-as}$
32. $\mathcal{L}\left\{(-1)^{[|t/a|]} \quad (\text{square wave})\right\} = \dfrac{1}{s}\tanh\left(\dfrac{as}{2}\right)$
33. $\mathcal{L}\left\{\left[\left|\dfrac{t}{a}\right|\right] \ (\text{staircase})\right\} = \dfrac{e^{-as}}{s\left(1-e^{-as}\right)}$

Miscellaneous Functions

1. $e^{ix} = \cos x + i\sin x$
2. $\sinh x = \dfrac{e^x - e^{-x}}{2}$
3. $\cosh x = \dfrac{e^x + e^{-x}}{2}$
4. $\tanh x = \dfrac{\sinh x}{\cosh x}$
5. $\operatorname{arctanh} u = \dfrac{1}{2}\ln\left|\dfrac{1+u}{1-u}\right|, \quad -1 < u < 1$

Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right] \text{ where } \begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\,dt \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\cos\dfrac{n\pi t}{L}\,dt \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\sin\dfrac{n\pi t}{L}\,dt \end{cases}$$